Higher Order Wall Boundary Conditions for Incompressible Flow Simulations

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Abstract

In this paper, the new higher order wall boundary conditions are proposed for solving the incompressible flows. The square driven cavity flows are simulated by using the variable order method of lines with the present wall boundary conditions. The variable order method of lines is constructed by the spatial discretization, i.e., the variable order proper convective scheme for convective terms and the modified differential quadrature method for diffusive terms, and time integration. The 2nd, 4th, 6th, and 8th order solutions are presented and these results show this higher order boundary conditions are very promising for the incompressible flow simulations.

Keyword: *Wall boundary condition, higher order solution, variable order method of lines*

1. Introduction

Today's incompressible flow simulations based on the finite difference method will be successfully applicable to the various practical flows by using DNS, LES and RANS. In the more practical flow simulations, however, the enormous computational time and memory are necessary in order to obtain the reliable solution. To relax the restriction on the limit of grid resolution, i.e., number of grid points, the higher order numerical simulation is one of the means of solving. In the higher order flow simulation, the boundary conditions, especially wall boundary conditions, with higher order of spatial accuracy are the more important problem. One of boundary treatments is the use of one-side finite difference. In this case, the programming is more complicated because the various finite difference formulas have to be adopted, and the compatibility condition for the pressure usually is not satisfied.

In this paper, the new higher order wall boundary conditions that introduce the virtual grid points inside the wall according to the stencil of higher order finite difference approximation, are proposed. In the present boundary conditions, the finite difference formulas at the near boundary are the same as those at the inner region. The square driven cavity flows are considered and the present higher order wall boundary conditions are verified.

2. Computational Method

2.1 Higher order flow solver

In this paper, we consider the incompressible flows. Then, the incompressible Navier-Stokes equations can be written by

$$\frac{\partial u_i}{\partial x_i} = 0 \quad , \tag{1}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j^{\,2}} \quad . \tag{2}$$

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Equations (1) and (2) denote the continuity equation and the momentum equations. The velocity components and the pressure are expressed by u_i and p, respectively. Re denotes the Reynolds number defined by $Re=UL/\nu$, where ν is the kinematic viscosity. The equations are nondimensionalized by the reference length L, the reference velocity U and the reference pressure ρU^2.

The variable order method of lines [1,2] is adopted for solving the incompressible Navier-Stokes equations. In the method of lines approach, the spatial derivatives are discretized by the appropriate scheme, so that the partial differential equations (PDEs) in space and time are reduced to the system of ordinary differential equations (ODEs) in time. The resulting ODEs are integrated by the Runge-Kutta type time integration scheme.

In the spatial discretization, the convective terms are approximated by the variable order proper convective scheme [3], because of the consistency of the discrete continuity equation, the conservation property, and the variable order of spatial accuracy. This scheme is the extension of the proper convective scheme proposed by Morinishi [4] to the variable order. The variable order proper convective scheme can be described by

$$u_j \frac{\partial u_i}{\partial x_j}\Big|_x = \sum_{\ell=1}^{M/2} c_{\ell'} \overline{\overline{u_j}^{x_j} \frac{\delta_{\ell'} u_i}{\delta_{\ell'} x_j}}^{\ell' x_j}\Big|_x \quad , \quad (\ell' = 2\ell - 1) \tag{3}$$

where M denotes the order of spatial accuracy, and the operators in Eq. (3) are defined by

$$\overline{f}^{\ell' x_j}\Big|_x = \frac{1}{2}(f_{x_j+\ell'/2} + f_{x_j-\ell'/2}) \quad , \tag{4}$$

$$\overline{u_j}^{x_j}\Big|_{x_j \pm \ell'/2} = \sum_{m=1}^{M/2} \frac{c_{m'}}{2}[u_j|_{x_j \pm (\ell'+m')/2} + u_j|_{x_j \pm (\ell'-m')/2}] \quad , \tag{5}$$

$$\frac{\delta_{\ell'} u_i}{\delta_{\ell'} x_j}\Big|_{x_j \pm \ell'/2} = \frac{\pm 1}{\ell' \Delta x_j}(u_i|_{x_j \pm \ell'} - u_i|_{x_j}) \quad , \tag{6}$$

where $m'=2m-1$. In this scheme, we can obtain the arbitrary order of spatial accuracy by changing only one parameter M. The coefficients $c_{\ell'}$ and $c_{m'}$ are the weighting parameters and Δx_j denotes the grid spacing in the x_j direction.

On the other hand, the diffusive terms are discretized by the modified differential quadrature (MDQ) method [5]. In the MDQ method, the spatial derivatives are approximated by the linear combination of the appropriate coefficients and the function itself. The second derivative can be discretized by

$$\frac{\partial^2 u_i}{\partial x_i^2}\Big|_x = \sum_{m=-M/2}^{M/2} \Phi_m''(x) u_i|_{x_i+m} \quad . \tag{7}$$

The coefficient $\Phi_m''(x)$ is the second derivatives of the function defined by

$$\Phi_m(x) = \frac{\Pi(x)}{(x - x_{i+m})\Pi'(x_{i+m})} \quad , \tag{8}$$

$$\Pi(x) = (x - x_{i-M/2}) \cdots (x - x_i) \cdots (x - x_{i+M/2}) \quad . \tag{9}$$

The coefficients of the variable order proper convective scheme, $c_{\ell'}$, can be determined automatically by using the MDQ coefficients for the first derivatives. For example, these coefficients can be determined as follows;

$c_1 = 1$ (2nd order)

$c_1 = 9/8,\ c_3 = -1/8$ (4th order)

$c_1 = 150/128,\ c_3 = -25/128,\ c_5 = 3/128$ (6th order)

$c_1 = 1225/1024,\ c_3 = -245/1024,\ c_5 = 49/1024,\ c_7 = -5/1024$ (8th order)

$c_1 = 39690/32768,\ c_3 = -8820/32768,\ c_5 = 2268/32768,\ c_7 = -405/32768,\ c_9 = 35/32768$ (10th order)

Then the partial differential equations in space and time are reduced to the system of ordinary differential equations (ODEs) in time. The resulting ODEs in time are integrated by the appropriate time integration scheme, e.g., the Runge-Kutta type scheme. In this paper, the collocated grid system which all variables are defined at cell center is adopted. The fractional step technique [6] is used for the solution procedure. In the first step, the fractional step velocity u_i* is computed by the relation,

$$u_i^* = u_i^n + \gamma \Delta t F_i^n \ , \tag{10}$$

where F_i denotes the convective and diffusive terms, and γ is the coefficient determined by the time integration scheme. The superscript n and * denote the values at $t = n\Delta t$ and the fractional step values, respectively. Equation (10) is solved on the collocated grid points. The computed fractional step velocity is interpolated into the staggered locations so that the staggered fractional step velocity, $\overline{u_i}^*$, can be obtained. By using Eq. (5), the variable order interpolation is constructed. The staggered velocity at next time step is given by the relation,

$$\overline{u_i}^{n+1} = \overline{u_i^*} - \gamma \Delta t \frac{\partial p^{n+1}}{\partial x_i} \ . \tag{11}$$

Substituting Eq. (11) into the discrete continuity equation, the pressure equation

$$\frac{\partial^2 p^{n+1}}{\partial {x_i}^2} = \frac{1}{\gamma \Delta t} \frac{\partial \overline{u_i^*}}{\partial x_i} \ , \tag{12}$$

can be obtained. This pressure equation is solved by the variable order SOR method. Finally, the velocity at next time step is computed by

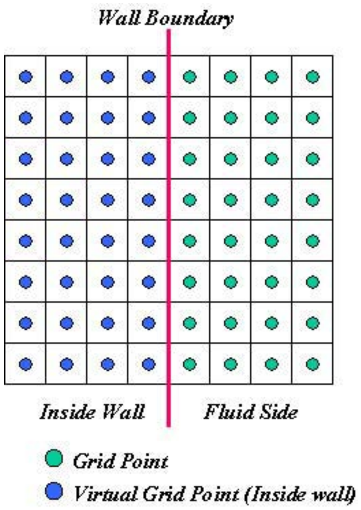
$$u_i^{n+1} = u_i^* - \gamma \Delta t \frac{\partial p^{n+1}}{\partial x_i} \ . \tag{13}$$

In practice, the higher order time integration scheme, e.g., 3rd or 4th order Runge-Kutta type scheme, should be adopted. In this paper, however, the forward Euler time integration scheme ($\gamma = 1$) is used, because we consider only the steady flow problem.

2.2 Higher order wall boundary conditions

In the collocated grid system, the wall boundary location is different from the grid point, that is, cell center, shown in Fig. 1. Then, in order to estimate the velocities on the virtual grid points inside the wall, it is necessary that the interpolated velocities are equivalent to the wall (boundary) velocities.

Generally, the interpolation is defined by the linear combination of the simple averages and coefficients.



$$f_i = \sum_{\ell=1}^{M/2} c_{\ell'} \overline{f_i}^{\ell'} \quad , \quad \overline{f_i}^{\ell'} = \frac{1}{2}(f_{i-\ell'/2} + f_{i+\ell'/2}) \quad , \tag{14}$$

where M and $c_{\ell'}$ are the same ones in Eq. (3) and $\ell' = 2\ell - 1$. The coefficients satisfy the relation,

$$\sum_{\ell=1}^{M/2} c_{\ell'} = 1 \tag{15}$$

By using this relation, the velocity, e.g., u, on the virtual grid point is specified by

$$\overline{u}^{\ell'} = \frac{1}{2}(u_{i-\ell'/2} + u_{i+\ell'/2}) = u_{wall} \tag{16}$$

Then, the interpolated velocity at the wall boundary satisfies.

d grid system

$$u_{wall} = \sum_{\ell=1}^{M/2} c_{\ell'} \overline{u}^{\ell'} \quad . \tag{17}$$

On the other hand, the first derivative of pressure at the boundary point is necessary to solve the pressure equation. The pressure on the virtual boundary point has to be specified by the first derivatives of pressure. The first derivative can be estimated by the linear combination of the second order finite difference with different grid spacings and coefficients.

$$\frac{\partial f}{\partial x_i} = \sum_{\ell=1}^{M/2} c_{\ell'} \frac{\delta_{\ell'} f}{\delta_{\ell'} x_i} \quad , \quad \frac{\delta_{\ell'} f}{\delta_{\ell'} x_i} = \frac{1}{\ell' \Delta x_i}(-f_{x_i-\ell'/2} + f_{x_i+\ell'/2}) \quad , \tag{18}$$

where Δx_i is the grid spacing in the x_i direction. Then, we set that the each second order finite difference is equal to the boundary value,

$$\frac{\delta_{\ell'} p}{\delta_{\ell'} x_i} = \frac{1}{\ell' \Delta x_i}(-p_{x_i-\ell'/2} + p_{x_i+\ell'/2}) = F_{wall} \quad , \tag{19}$$

so that the global Neumann condition can be satisfied,

$$\frac{\partial p}{\partial x_i} = \sum_{\ell=1}^{M/2} c_{\ell'} \frac{\delta_{\ell'} p}{\delta_{\ell'} x_i} = F_{wall} \quad , \tag{20}$$

where F_{wall} denotes the convective and diffusive terms at the wall boundary.

Moreover, in the case computing the second derivative of velocity at the boundary, the velocities on half grid location shown in Fig. 2 are necessary. In this paper, these velocities are determined by using the global conservation [7]. The global conservation is that the mass conservation has to be specified only by the inflow and outflow at the both boundaries, e.g., when we consider the x direction, the global conservation can be written by

$$\sum_{i=1}^{N} \Delta x \frac{\partial u}{\partial x}|_i = -u_{1/2} + u_{N+1/2} \quad , \tag{21}$$

where N is the number of grid points in the x direction. The subscripts $1/2$ and $N+1/2$ denote the left and right boundary values, respectively. In the 2nd order of spatial accuracy, the above global conservation is satisfied automatically, because of

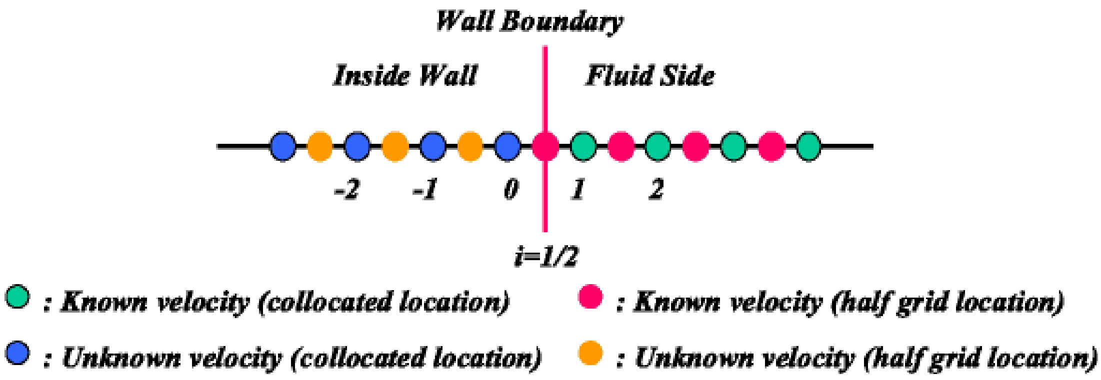


Fig. 2 Grid points location

$$\frac{\partial u}{\partial x}|_i^{2nd} = \frac{1}{\Delta x}(-u_{i-1/2} + u_{i+1/2}) \quad . \tag{22}$$

However, it is not satisfied generally in the higher order of spatial accuracy. For example, the global conservation in the 4th order is written by

$$\begin{aligned} \sum_{i=1}^{N} \Delta x \frac{\partial u}{\partial x}|_i^{4th} &= c_1(-u_{1/2} + u_{N+1/2}) \\ &+ \frac{c_3}{3}(-u_{-1/2} - u_{1/2} - u_{3/2} + u_{N-1/2} + u_{N+1/2} + u_{N+3/2}) \quad . \end{aligned} \tag{23}$$

Equation (23) does not satisfy Eq. (21) automatically. Then, in order to coincide Eq. (21) with Eq. (23), we need the following relation,

$$\begin{aligned} &\frac{1}{3}(-u_{-1/2} - u_{1/2} - u_{3/2}) = -u_{1/2} \quad , \\ &\frac{1}{3}(u_{N-1/2} + u_{N+1/2} + u_{N+3/2}) = u_{N+1/2} \quad . \end{aligned} \tag{24}$$

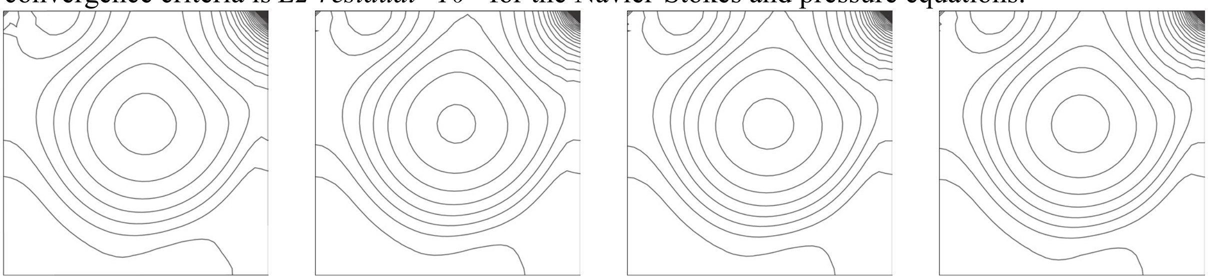
Therefore, the velocity components at half grid location inside the wall, $u_{-1/2}$ and $u_{N+3/2}$, can be estimated. And in order to compute the second derivatives at the boundary by usual centered finite difference, the velocity components, $u_{-3/2}$ and $u_{N+5/2}$, have to be determined. These can be specified by considering the continuity equation at the closest cell inside the wall to the boundary. In the case of Fig. 2, the following relation can be obtained from the continuity equation on $(0,j)$.

$$\frac{\partial u}{\partial x}\Big|_{0,j} = \frac{c_1}{\Delta x}(-u_{-1/2,j} + u_{1/2,j}) + \frac{c_3}{3\Delta x}(-u_{-3/2,j} + u_{3/2,j}) = -\frac{\partial v}{\partial y}\Big|_{0,j} \tag{25}$$

The right hand side of Eq. (25) is known by using the interpolated values. Then, the velocity, $u_{-3/2}$, can be estimated. Similarly, the higher order wall boundary conditions than the 4th order of spatial accuracy can be formulated.

3. Computational Results

In this paper, the square driven cavity flow problem is considered. The computational conditions are that the number of grid points is *41x41* (uniform grid), Reynolds number *Re=1000* and *5000*, and the convergence criteria is *L2-residual*$<10^{-6}$ for the Navier-Stokes and pressure equations.



(a) 2nd order solution (b) 4th order solution (c) 6th order solution (d) 8th order solution

Fig. 3. Pressure coefficient distributions (*Re=1000*)

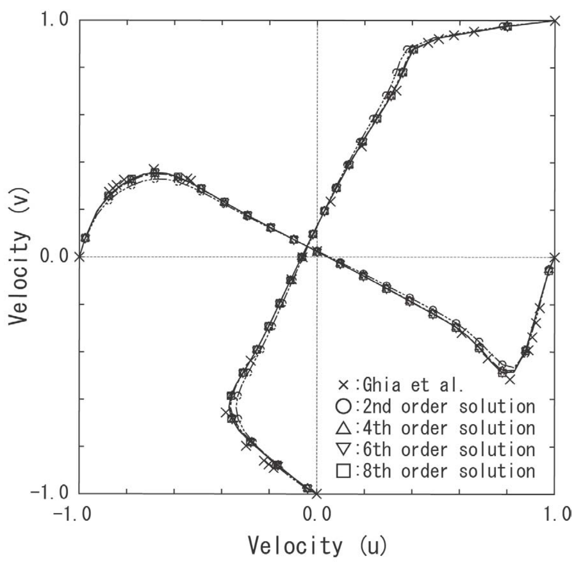


Fig. 4. Velocity profiles along the center lines (*Re=1000*, *41x41* grid points)

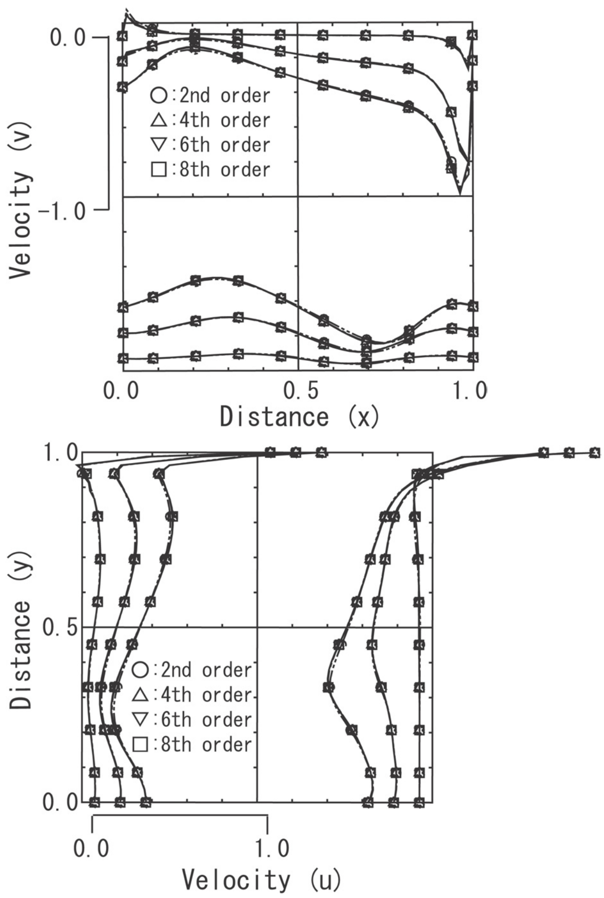


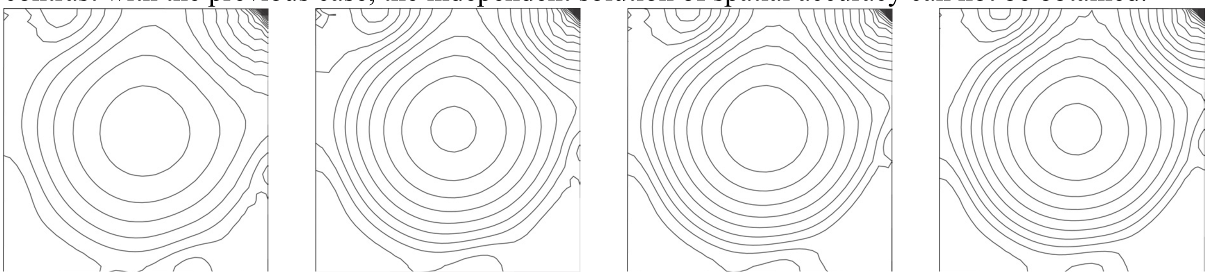
Fig. 5. Velocity profiles near the boundaries

Figures 3 and 4 show the pressure coefficient distributions and the velocity profiles along the center lines with the Reynolds number $Re=1000$ and, the 2nd, 4th, 6th, and 8th order of spatial accuracy, respectively. The pressure coefficient is defined by

$$Cp = \frac{p - p_0}{1/2\rho U^2} \quad . \tag{26}$$

where p_0 is the pressure at center of bottom cavity wall and U denotes the moving wall velocity. In Fig. 4, the cross symbol denotes the result with *129x129* grid points obtained by Ghia et al. [8]. The velocity profiles near the boundaries are plotted in Fig. 5. The circular, triangular, inverse triangular, and square symbols denote the 2nd, 4th, 6th, and 8th order solutions, respectively. In this Reynolds number, the higher order solutions than the second order are almost the same. In comparison with the Ghia's solution, these higher order velocity profiles are clearly improved.

Figures 6, 7, and 8 show the results in Reynolds number $Re=5000$, that is, the pressure coefficient distributions, velocity profiles along the center lines, and velocity profiles near the boundaries. In contrast with the previous case, the independent solution of spatial accuracy can not be obtained.



(a) 2nd order solution (b) 4th order solution (c) 6th order solution (d) 8th order solution

Fig. 6. Pressure coefficient distributions ($Re=5000$)

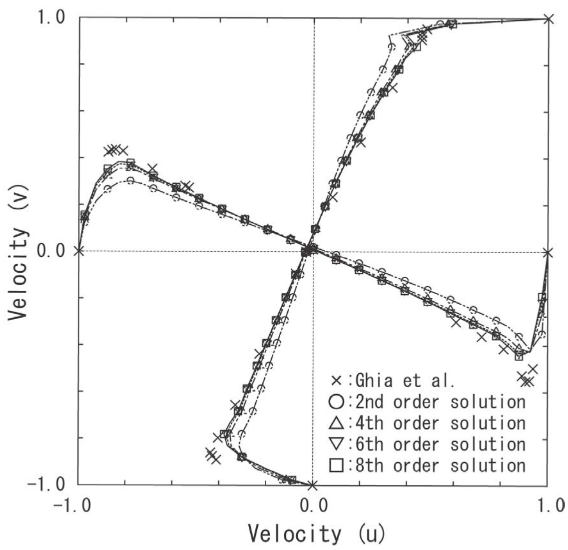


Fig. 7. Velocity profiles along the center lines ($Re=5000$, $41x41$ grid points)

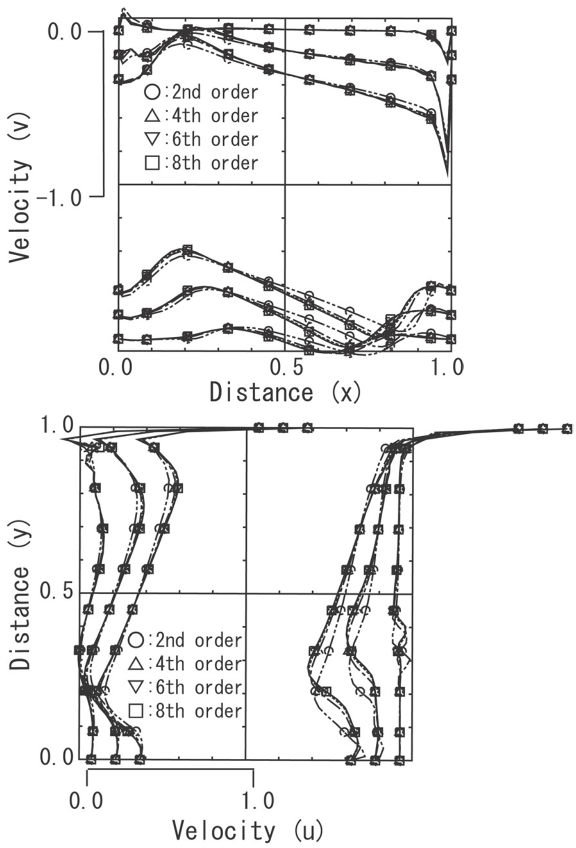


Fig. 8. Velocity profiles near the boundaries

However, as the spatial accuracy becomes higher, the primary vortex becomes stronger and the velocity profiles become closer to the Ghia's solution. Especially, the velocity profiles near the boundaries are different between the orders of spatial accuracy. These distributions show the tendency to converge at the highest order solution. Then, the present higher order wall boundary conditions are effective to improve the solutions.

4. Concluding Remarks

In this paper, the higher order wall boundary conditions for incompressible flow simulation are presented on the collocated grid system. The validation is performed in the steady square driven cavity flow problems, i.e., $Re=1000$ and *5000* with *41x41* grid points, by using the variable order method of lines.

In the case of the Reynolds number $Re=1000$, the independent solution of spatial accuracy can be obtained. On the other hand, as the order of spatial accuracy becomes higher, the primary vortex becomes stronger and the velocity components near the boundaries are clearly improved in the case of the Reynolds number $Re=5000$.

Then, it is concluded that the present higher order wall boundary conditions are more available for the incompressible flow simulations. In this paper, the higher order boundary conditions are constructed on the collocated grid system, but it is possible to build the almost the same boundary conditions on the staggered grid system. Therefore, this idea of higher order boundary conditions is very promising for the higher order incompressible flow simulations.

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